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Constants of Strong Unicity of Minimal Projections onto some Two-Dimensional Subspaces of $l_\infty^{(4)}$

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In this paper the constants of strong unicity of minimal projections onto some two-dimensional subspaces in $l_\infty^{(4)}$ will be calculated.

1. INTRODUCTION

Let X be a normed space and let Y be a linear subspace of X. A bounded linear operator $\pi : X \to Y$ is called a projection if $\pi y = y$ for any $y \in Y$. The set of all projections going from X onto Y will be denoted by $\pi(X, Y)$. Set $\lambda(Y, X) = \inf\{||\pi|| : \pi \in \pi(X, Y)\}$. A projection π_0 is minimal if $||\pi_0|| = \lambda(Y, X)$. This projection π_0 is called strongly unique if there is $k \in (0; 1]$ such that for any $\pi \in \pi(X, Y)$,

$$||\pi_0|| + k||\pi - \pi_0|| \leqslant ||\pi||.$$

The study of existence and unicity of minimal projections is related to the study of best approximation.

Let $X = l_\infty^{(n)}$ and let $Y = Y_{n-2}$, where $Y_{n-2} \subset l_\infty^{(n)}$ is a subspace of codimension two. Then for any $\pi \in \pi(l_\infty^{(n)}, Y_{n-2})$,

$$\pi_{\alpha,\beta} x = x - \alpha f(x) - \beta g(x),$$

where $\alpha \in l_\infty^{(n)}$, $\beta \in l_\infty^{(n)}$, f and g are two linear functionals defined on $l_\infty^{(n)}$, and

$$f(\alpha) = g(\beta) = 1, \qquad f(\beta) = g(\alpha) = 0. \tag{1}$$

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If moreover $f^{-1}(0)=\{x\in l_\infty^{(n)}|f(x)=\sum_{i=1}^n f_ix_i=0\}$, $g^{-1}(0)=\{x\in l_\infty^{(n)}|\,g(x)=\sum_{i=1}^n g_ix_i=0\}$ are hyperplanes in $l_\infty^{(n)}$, then we can write $Y_{n-2}=f^{-1}(0)\cap g^{-1}(0)$. The norms of projections π and $\pi-\pi_0$ calculated by formulas

$$||\pi||=\max_{1\leqslant i\leqslant n} T_i, \qquad \text{where } T_i=\sum_{j=1}^n |\delta_{ij}-\alpha_if_j-\beta_ig_j|$$

and

$$||\pi-\pi_0||=\max_{1\leqslant i\leqslant n} B_i, \qquad \text{where } B_i=\sum_{j=1}^n |(\alpha_i-\alpha_i^{(0)})f_j+(\beta_i-\beta_i^{(0)})g_j|.$$

For projection π_0 we have

$$\pi_{\alpha,\beta}^{(0)}x=x-\alpha^{(0)}f(x)-\beta^{(0)}g(x).$$

For more complete information about this subject the reader is referred to [1–13]. In [2, 5, 7–13] a complete characterization and unicity of minimal projections on hyperplanes and subspaces of codimension two in spaces l_1, c_0 and their finite-dimensional analogues $l_1^{(n)}, l_\infty^{(n)}$ are presented. The strongly unique minimal projections on hyperplanes in $l_\infty^{(n)}$ and $l_1^{(n)}$ and the same onto two-dimensional subspaces of $l_\infty^{(4)}$ are considered in [1, 3–6].

In this paper we calculate the constants of strong unicity for some two-dimensional subspaces in $l_\infty^{(4)}$.

2. THE CONSTANTS OF STRONG UNICITY

Let functionals f and g be of the form

$$f=(1,s,r,0), \qquad g=(0,0,0,1), \tag{2}$$

where parameters $s>0, r>0$. Then conditions (1) can be rewritten in the form

$$f(\alpha)=\alpha_1+s\alpha_2+r\alpha_3=1, \qquad g(\alpha)=\alpha_4=0,$$

$$f(\beta)=\beta_1+s\beta_2+r\beta_3=0, \qquad g(\beta)=\beta_4=1.$$

By Theorem 3.3 from [6] it is easy to deduce the following (see also [5, Theorem 2.4.6, p. 73; 4, Theorem 2.5]):

LEMMA 2.1. *Let* $Y_2 \subset l_\infty^{(4)}$ *be a subspace of codimension two,* $Y_2 = f^{-1}(0) \cap g^{-1}(0)$, *where* f *and* g *are functionals* (2), *and let* $\pi_{\alpha,\beta}^{(0)}$ *be a minimal projection on subspace* Y_2. *Then* (1) $\|\pi_{\alpha,\beta}^{(0)}\| = \frac{4rs}{Q}$, *where* $Q = 1 + (s + r - 2)(s - r)^2 + (s + r - 1)(s + r)$, *if* $s + r - 1 > 0$, $r \leqslant s \leqslant 1$; (2) $\|\pi_{\alpha,\beta}^{(0)}\| = 1$, if $1 - s - r \geqslant 0$, $r \leqslant s$.

Now we can prove the main result of this paper. First note that in [3, Theorem III.3.1, p. 105] the strong unicity constant has been estimated for minimal projections onto hyperplanes in $l_\infty^{(n)}$.

THEOREM 2.2. *Let projection* $\pi_{\alpha,\beta}^{(0)}$ *and functionals* f, g *be as in Lemma* 2.1. *Then the projection* $\pi_{\alpha,\beta}^{(0)}$ *is strongly unique* (*it follows from* [4, *Theorems* 3.1, 3.3, 3.4] (*see also* [5, *Theorems* 2.5.1, 2.5.2, 2.5.3, pp. 75–78])) *and the* constant k *of strong unicity is equal to*

$$k = \frac{r(1 - s + r)(s + r - 1)}{((1 - s)^2 + r(1 + s))(1 + s + r)} \qquad \text{if } s + r - 1 > 0, \quad r \leqslant s \leqslant 1$$

and

$$k = \frac{1 - s + r}{1 + s + r} \qquad \text{if } 1 - s - r > 0, \quad r \leqslant s.$$

Proof. Let

$$s + r - 1 > 0, \qquad r \leqslant s \leqslant 1. \tag{3}$$

In this case we have

$$T_1 = |1 - \alpha_1| + s|\alpha_1| + r|\alpha_1| + |\beta_1| \geqslant 1 + (s + r - 1)|\alpha_1| + |\beta_1|$$
$$\geqslant 1 + (s + r - 1)\alpha_1,$$

$$T_2 = |\alpha_2| + |1 - s\alpha_2| + r|\alpha_2| + |\beta_2| \geqslant 1 + (1 - s + r)|\alpha_2| + |\beta_2|$$
$$\geqslant 1 + (1 - s + r)\alpha_2,$$

$$T_3 = |\alpha_3| + s|\alpha_3| + |1 - r\alpha_3| + |\beta_3| \geqslant 1 + (1 + s - r)|\alpha_3| + |\beta_3|$$
$$\geqslant 1 + (1 + s - r)\alpha_3,$$

$$T_4 = |\alpha_4| + s|\alpha_4| + r|\alpha_4| + |1 - \beta_4| = 0.$$

Now we find $\alpha_i^{(0)}$, $\beta_i^{(0)}$ such that $T_i = ||\pi_{\alpha,\beta}^{(0)}||$ $(i = 1, 2, 3)$. Let $\beta_i^{(0)} = 0$ $(i = 1, 2, 3)$. It is obtained above that $\alpha_4^{(0)} = 0$, $\beta_4^{(0)} = 1$. To find $\alpha_i^{(0)}$ $(i = 1, 2, 3)$ consider a system of equations

$$\begin{cases} 1 + (s + r - 1)\alpha_1^{(0)} = \frac{4rs}{Q}, \\ 1 + (1 - s + r)\alpha_2^{(0)} = \frac{4rs}{Q}, \\ 1 + (1 + s - r)\alpha_3^{(0)} = \frac{4rs}{Q}. \end{cases}$$

From this system we get

$$\alpha_1^{(0)} = \frac{(1 - s + r)(1 + s - r)}{Q}, \qquad \alpha_2^{(0)} = \frac{(1 + s - r)(s + r - 1)}{Q},$$

$$\alpha_3^{(0)} = \frac{(1 - s + r)(s + r - 1)}{Q}.$$

It is easy to show that $\alpha_i^{(0)}$ $(i = 1, 2, 3)$ satisfy the condition $f(\alpha) = 1$.

Calculate the norm of the operator $\pi - \pi_{\alpha,\beta}^{(0)}$:

$$\begin{aligned} ||\pi - \pi_{\alpha,\beta}^{(0)}|| &= \max_{1 \leqslant i \leqslant 4} B_i = \max_i \sum_{j=1}^{4} |(\alpha_i - \alpha_i^{(0)})f_j + (\beta_i - \beta_i^{(0)})g_j| \\ &= \max\left(\sum_{j=1}^{4}\left|\left(\alpha_1 - \frac{(1 - s + r)(1 + s - r)}{Q}\right)f_j + (\beta_1 - \beta_1^{(0)})g_j\right|;\right. \\ &\quad \sum_{j=1}^{4}\left|\left(\alpha_2 - \frac{(1 + s - r)(s + r - 1)}{Q}\right)f_j + (\beta_2 - \beta_2^{(0)})g_j\right|; \\ &\quad \left.\sum_{j=1}^{4}\left|\left(\alpha_3 - \frac{(1 - s + r)(s + r - 1)}{Q}\right)f_j + (\beta_3 - \beta_3^{(0)})g_j\right|; 0\right) \\ &= \max\left\{\left|\alpha_1 - \frac{(1 - s + r)(1 + s - r)}{Q}\right|(1 + s + r) + |\beta_1|;\right. \\ &\quad \left|\alpha_2 - \frac{(1 + s - r)(s + r - 1)}{Q}\right|(1 + s + r) + |\beta_2|; \\ &\quad \left.\left|\alpha_3 - \frac{(1 - s + r)(s + r - 1)}{Q}\right|(1 + s + r) + |\beta_3|\right\}. \end{aligned}$$

Put $0<\alpha_3<\frac{(1-s+r)(s+r-1)}{Q}$, $\beta_i=0$ $(i=1,2,3)$. Moreover, let

$$\alpha_1>\frac{(1-s+r)(1+s-r)}{Q},\qquad \alpha_2>\frac{(1+s-r)(s+r-1)}{Q}. \tag{4}$$

Applying conditions (4) it is easy to show that if $\alpha_3<\alpha_3^{(0)}$, then the inequality $\alpha_1+s\alpha_2>1-r\alpha_3^{(0)}$ is satisfied.

Put also $1+(s+r-1)\alpha_1=1+(1-s+r)\alpha_2$, hence

$$\alpha_2=\frac{s+r-1}{1-s+r}\alpha_1. \tag{5}$$

By condition $f(\alpha)=1$ we get

$$\alpha_1+s\alpha_2=1-r\alpha_3. \tag{6}$$

Applying conditions (5) and (6) find

$$\alpha_1=\frac{1-s+r}{P}(1-r\alpha_3),\qquad \alpha_2=\frac{s+r-1}{P}(1-r\alpha_3),$$
$$\text{where } P=(1-s)^2+r(1+s).$$

For these values of α_i and β_i we get

$$B_1=\left|\alpha_1-\frac{(1-s+r)(1+s-r)}{Q}\right|(1+s+r)$$
$$=\frac{r(1-s+r)(1+s+r)}{P}\left(\frac{(1-s+r)(s+r-1)}{Q}-\alpha_3\right),$$

$$B_2=\left|\alpha_2-\frac{(1+s-r)(s+r-1)}{Q}\right|(1+s+r)$$
$$=\frac{r(s+r-1)(1+s+r)}{P}\left(\frac{(1-s+r)(s+r-1)}{Q}-\alpha_3\right),$$

$$B_3=(1+s+r)\left(\frac{(1-s+r)(s+r-1)}{Q}-\alpha_3\right).$$

The inequality $B_1\geqslant B_2$ is equivalent to $1-s+r\geqslant s+r-1$, which immediately follows from (3). Now we prove that $B_3>B_1$. We have

$$1+s+r>\frac{r(1-s+r)(1+s+r)}{P},$$

hence

$$(1-s)^2 + 2rs > r^2. \tag{7}$$

Since $(1-s)^2 + 2rs \geqslant 2rs \geqslant 2r^2 > r^2$, the last inequality is satisfied. Consequently, $\max_{1\leqslant i\leqslant 3} B_i = B_3$.

Moreover, $T_1 = T_2 = 1 + \frac{(1-s+r)(s+r-1)}{P}(1 - r\alpha_3)$, $T_3 = 1 + (1+s-r)\alpha_3$. Since inequality $T_3 < T_1$ is equivalent to $\alpha_3 < \frac{(1-s+r)(s+r-1)}{Q}$ we obtain $\max_{1\leqslant i\leqslant 3} T_i = T_1$.

Now we will estimate k from above. From the inequality $||\pi^{(0)}_{\alpha,\beta}|| + kB_3 \leqslant T_1$, by elementary calculations, we may get $k \leqslant \frac{r(1-s+r)(s+r-1)}{(1+s+r)P}$. Now we show that $k \in (0,1]$. Obviously, $k > 0$ if conditions (3) are satisfied. The condition $k \leqslant 1$ follows from the two inequalities $\frac{r(1-s+r)}{P} < 1$ and $\frac{s+r-1}{1+s+r} \leqslant 1$. The first inequality is equivalent to (7). The second inequality is trivial.

To show that $k = \frac{r(1-s+r)(s+r-1)}{(1+s+r)P}$ is a maximal value of the constant of a strong unicity we prove that inequality

$$\begin{aligned}
&||\pi^{(0)}_{\alpha,\beta}|| + k \max_{1\leqslant i\leqslant 3}\{|\alpha_i - \alpha^{(0)}_i|(1+s+r) + |\beta_i|\} \\
&\quad\leqslant \max\{1 + (s+r-1)|\alpha_1| + |\beta_1|;\ 1 + (1-s+r)|\alpha_2| \\
&\qquad + |\beta_2|;\ 1 + (1+s-r)|\alpha_3| + |\beta_3|\}
\end{aligned} \tag{8}$$

satisfies for any α_i, β_i.

Consider three cases.

(1) Let $\max_i B_i = B_1$. Denote $\alpha_1 - \alpha^{(0)}_1 = \varepsilon_1$.

(a) Suppose that $\varepsilon_1 \geqslant 0$. Then (8) can be rewritten in the form $||\pi^{(0)}_{\alpha,\beta}|| + k(\varepsilon_1(1+s+r) + |\beta_1|) \leqslant 1 + (s+r-1)|\alpha_1| + |\beta_1|$, hence $\frac{4rs}{Q} - 1 + k(\varepsilon_1(1+s+r) + |\beta_1|) \leqslant (s+r-1)\left(\varepsilon_1 + \frac{(1-s+r)(1+s-r)}{Q}\right) + |\beta_1|$, then $k(\varepsilon_1(1+s+r) + |\beta_1|) \leqslant (s+r-1)\varepsilon_1 + |\beta_1|$.

The last inequality is equivalent to $k\varepsilon_1(1+s+r) \leqslant (s+r-1)\varepsilon_1$ and $k|\beta_1| \leqslant |\beta_1|$, which immediately follow from (7) and $k \leqslant 1$ accordingly.

(b) Now assume that $\varepsilon_1 < 0$. To prove (8) we note that $\max\{1 + (s+r-1)|\alpha_1| + |\beta_1|;\ 1 + (1-s+r)|\alpha_2| + |\beta_2|;\ 1 + (1+s-\dot{r})|\alpha_3| + |\beta_3|\} \geqslant \lambda_2(1 + (1-s+r)|\alpha_2| + |\beta_2|) + \lambda_3(1 + (1+s-r)|\alpha_3| + |\beta_3|)$, where $\lambda_2 \geqslant 0$, $\lambda_3 \geqslant 0$, $\lambda_2 + \lambda_3 = 1$.

In this case (8) is equivalent to

$$||\pi^{(0)}_{\alpha,\beta}|| + k(1+s+r)|\varepsilon_1| \leqslant \lambda_2(1 + (1-s+r)|\alpha_2|) + \lambda_3(1 + (1+s-r)|\alpha_3|) \tag{9}$$

and

$$k|\beta_1|\leqslant\lambda_2|\beta_2|+\lambda_3|\beta_3|. \tag{10}$$

To prove these inequalities put $\lambda_2=\frac{s\mu}{1-s+r}$, $\lambda_3=\frac{r\mu}{1+s-r}$. By condition $\lambda_2+\lambda_3=1$ we find that $\mu=\frac{(1-s+r)(1+s-r)}{R}$, where $R=(s-r)^2+s+r$.

By condition $f(\alpha)=1$, (9) can be rewritten in the form

$||\pi^{(0)}_{\alpha,\beta}||+k(1+s+r)|\varepsilon_1|\leqslant 1+\mu(1-\alpha_1)$, hence $||\pi^{(0)}_{\alpha,\beta}||+k(1+s+r)|\varepsilon_1|\leqslant 1+\mu(1-\alpha_1^{(0)}-\varepsilon_1)$, then $\frac{(s+r-1)(1+s-r)}{Q}+\frac{r(s+r-1)}{P}|\varepsilon_1|\leqslant\frac{1+s-r}{R}\left(\frac{(s+r-1)R}{Q}+|\varepsilon_1|\right)$. The last inequality is equivalent to $\frac{r(s+r-1)}{P}\leqslant\frac{1+s-r}{R}$, which follows from $s+r-1\leqslant 1+s-r$ and $rR\leqslant P$. The first inequality is equivalent to condition $r\leqslant 1$. The second inequality is equivalent to $(1+r-s)^2\geqslant 0$. Thus, inequality (9) is proved.

Now we prove (10). By condition $f(\beta)=0$ we get that $\beta_1=-s\beta_2-r\beta_3$, hence $|\beta_1|\leqslant s|\beta_2|+r|\beta_3|$. Instead of inequality (10) we prove that

$$k(s|\beta_2|+r|\beta_3|)\leqslant\frac{s(1+s-r)}{R}|\beta_2|+\frac{r(1-s+r)}{R}|\beta_3|,$$

which follows from two inequalities $k|\beta_2|\leqslant\frac{1+s-r}{R}|\beta_2|$ and $k|\beta_3|\leqslant\frac{1-s+r}{R}|\beta_3|$. The last inequalities immediately follow from $s+r-1<1+s-r, s+r-1<1+s+r$, $1-s+r<1+s+r$ and $rR\leqslant P$. Inequality (10) is proved too.

(2) Let $\max_i B_i=B_2$. Denote $\alpha_2-\alpha_2^{(0)}=\varepsilon_2$.

(a) Suppose that $\varepsilon_2\geqslant 0$. Inequality (8) can be rewritten in the form $||\pi^{(0)}_{\alpha,\beta}||+k(\varepsilon_2(1+s+r)+|\beta_2|)\leqslant 1+(1-s+r)|\alpha_2|+|\beta_2|$, hence $\frac{4rs}{Q}-1+k(\varepsilon_2(1+s+r)+|\beta_2|)\leqslant(1-s+r)\left(\varepsilon_2+\frac{(s+r-1)(1+s-r)}{Q}\right)+|\beta_2|$, then $k(\varepsilon_2(1+s+r)+|\beta_2|)\leqslant(1-s+r)\varepsilon_2+|\beta_2|$.

The last inequality is equivalent to $k\varepsilon_2(1+s+r)\leqslant(1-s+r)\varepsilon_2$ and $k|\beta_2|\leqslant|\beta_2|$. The first inequality is equivalent to $\frac{r(s+r-1)}{P}\leqslant 1$, which immediately follows from $(1-s)^2+r(2-r)\geqslant 0$. The second inequality is equivalent to $k\leqslant 1$.

(b) Now assume that $\varepsilon_2<0$. To prove (8), we use the inequality $\max\{1+(s+r-1)|\alpha_1|+|\beta_1|;\ 1+(1-s+r)|\alpha_2|+|\beta_2|;\ 1+(1+s-r)|\alpha_3|+|\beta_3|\}\geqslant\lambda_1(1+(s+r-1)|\alpha_1|+|\beta_1|)+\lambda_3(1+(1+s-r)|\alpha_3|+|\beta_3|)$, where $\lambda_1\geqslant 0$, $\lambda_3\geqslant 0$, $\lambda_1+\lambda_3=1$. Inequality (8) in this case follows from

$$||\pi^{(0)}_{\alpha,\beta}||+k(1+s+r)|\varepsilon_2|\leqslant\lambda_1(1+(s+r-1)|\alpha_1|)+\lambda_3(1+(1+s-r)|\alpha_3|), \tag{11}$$

$$k|\beta_2|\leqslant\lambda_1|\beta_1|+\lambda_3|\beta_3|. \tag{12}$$

To prove these inequalities put $\lambda_1 = \frac{\eta}{s+r-1}$, $\lambda_3 = \frac{r\eta}{1+s-r}$. By condition $\lambda_1 + \lambda_3 = 1$ find that $\eta = \frac{(s+r-1)(1+s-r)}{T}$, where $T = (1-r)^2 + s(1+r)$. By condition $f(\alpha) = 1$, (11) can be rewritten in the form $||\pi^{(0)}_{\alpha,\beta}|| + k(1+s+r)|\varepsilon_2| \leqslant 1 + \eta(1 - s\alpha_2)$, hence $||\pi^{(0)}_{\alpha,\beta}|| + k(1+s+r)|\varepsilon_2| \leqslant 1 + \eta(1 - s\alpha_2^{(0)} + |\varepsilon_2|)$, then $\frac{(1-s+r)(1+s-r)}{Q} + \frac{r(1-s+r)}{P}|\varepsilon_2| \leqslant \frac{1+s-r}{T}(\frac{(1-s+r)T}{Q} + s|\varepsilon_2|)$. The last inequality we rewrite in the form $\frac{r(1-s+r)}{P} \leqslant \frac{s(1+s-r)}{T}$. This inequality follows from $1-s+r \leqslant 1+s-r$ and $rT \leqslant sP$. The first inequality is equivalent to condition $r \leqslant s$. The second inequality is equivalent to $(s+r-1)^2 \geqslant 0$. Thus, (11) is proved.

Now we prove (12). From condition $f(\beta) = 0$ we get that $s\beta_2 = -\beta_1 - r\beta_3$, hence $s|\beta_2| \leqslant |\beta_1| + r|\beta_3|$. Instead of inequality (12) we prove that

$$\frac{k}{s}(|\beta_1| + r|\beta_3|) \leqslant \frac{1+s-r}{T}|\beta_1| + \frac{r(s+r-1)}{T}|\beta_3|,$$

which follows from two inequalities $k|\beta_1| \leqslant \frac{s(1+s-r)}{T}|\beta_1|$ and $k|\beta_3| \leqslant \frac{s(s+r-1)}{T}|\beta_3|$.

The last inequalities immediately follow from $1-s+r \leqslant 1+s-r$, $s+r-1 < 1+s+r$, $1-s+r < 1+s+r$ and $rT \leqslant sP$. Inequality (12) is proved too.

(3) Let $\max_i B_i = B_3$. Denote $\alpha_3 - \alpha_3^{(0)} = \varepsilon_3$.

(a) Suppose that $\varepsilon_3 \geqslant 0$. Inequality (8) can be rewritten in the form $||\pi^{(0)}_{\alpha,\beta}|| + k(\varepsilon_3(1+s+r) + |\beta_3|) \leqslant 1 + (1+s-r)|\alpha_3| + |\beta_3|$, hence $\frac{4rs}{Q} - 1 + k(\varepsilon_3(1+s+r) + |\beta_3|) \leqslant (1+s-r)(\varepsilon_3 + \frac{(s+r-1)(1-s+r)}{Q}) + |\beta_3|$, then $k(\varepsilon_3(1+s+r) + |\beta_3|) \leqslant (1+s-r)\varepsilon_3 + |\beta_3|$.

The last inequality follows from $k\varepsilon_3(1+s+r) \leqslant (1+s-r)\varepsilon_3$ and $k|\beta_3| \leqslant |\beta_3|$. The first inequality is equivalent to (7). The second inequality is equivalent to condition $k \leqslant 1$.

(b) Now assume that $\varepsilon_3 < 0$. To prove (8), we use the inequality $\max\{1 + (s+r-1)|\alpha_1| + |\beta_1|;\ 1 + (1-s+r)|\alpha_2| + |\beta_2|;\ 1 + (1+s-r)|\alpha_3| + |\beta_3|\} \geqslant \lambda_1(1 + (s+r-1)|\alpha_1| + |\beta_1|) + \lambda_2(1 + (1-s+r)|\alpha_2| + |\beta_2|)$, where $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$.

Inequality (8) in this case follows from

$$||\pi^{(0)}_{\alpha,\beta}|| + k(1+s+r)|\varepsilon_3| \leqslant \lambda_1(1 + (s+r-1)|\alpha_1|) + \lambda_2(1 + (1-s+r)|\alpha_2|), \tag{13}$$

$$k|\beta_3| \leqslant \lambda_1|\beta_1| + \lambda_2|\beta_2|. \tag{14}$$

To prove these inequalities put $\lambda_1 = \frac{\gamma}{s+r-1}, \lambda_2 = \frac{s\gamma}{1-s+r}$. By condition $\lambda_1 + \lambda_2 = 1$ we find that $\gamma = \frac{(s+r-1)(1-s+r)}{P}$.

By condition $f(\alpha)=1$, (13) can be rewritten in the form $||\pi^{(0)}_{\alpha,\beta}||+k(1+s+r)|\varepsilon_3|\leqslant 1+\gamma(1-r\alpha_3)$, hence $||\pi^{(0)}_{\alpha,\beta}||+k(1+s+r)|\varepsilon_3|\leqslant 1+\gamma(1-r\alpha^{(0)}_3+|\varepsilon_3|)$, then $\frac{1+s-r}{Q}+\frac{r}{P}|\varepsilon_3|\leqslant\frac{1}{P}(\frac{(1+s-r)P}{Q}+r|\varepsilon_3|)$, that is $\frac{r}{P}|\varepsilon_3|\leqslant\frac{r}{P}|\varepsilon_3|$. Thus, inequality (13) is proved.

Now we prove (14). From the condition $f(\beta)=0$ we get that $r\beta_3=-\beta_1-s\beta_2$, hence $r|\beta_3|\leqslant|\beta_1|+s|\beta_2|$. Instead of inequality (14) we prove that

$$\frac{k}{r}(|\beta_1|+s|\beta_2|)\leqslant\frac{1-s+r}{P}|\beta_1|+\frac{s(s+r-1)}{P}|\beta_2|,$$

which follows from the two inequalities $k|\beta_1|\leqslant\frac{r(1-s+r)}{P}|\beta_1|$ and $k|\beta_2|\leqslant\frac{r(s+r-1)}{P}|\beta_2|$. The last inequalities follow from the two trivial conditions $s+r-1<1+s+r,\ 1-s+r\leqslant 1+s+r$. Inequality (14) is proved.

The first requirement of Theorem 2.2. is proved too.

Now let

$$1-s-r\geqslant 0,\qquad r\leqslant s.$$

In this case we have $\alpha^{(0)}_1=1, \alpha^{(0)}_2=\alpha^{(0)}_3=\alpha^{(0)}_4=0,\ \beta^{(0)}_i=0\ (i=1,2,3)$, $\beta^{(0)}_4=1$. Calculate the norm of the operator $\pi-\pi^{(0)}_{\alpha,\beta}$:

$$\begin{aligned}||\pi-\pi^{(0)}_{\alpha,\beta}|| &= \max_{1\leqslant i\leqslant 4} B_i\\ &=\max\left(\sum_{j=1}^{4}|(\alpha_1-1)f_j+\beta_1 g_j|;\ \sum_{j=1}^{4}|\alpha_2 f_j+\beta_2 g_j|;\ \sum_{j=1}^{4}|\alpha_3 f_j+\beta_3 g_j|;\ 0\right)\\ &=\max\{|\alpha_1-1|(1+s+r)+|\beta_1|;\ |\alpha_2|(1+s+r)+|\beta_2|;\\ &\quad |\alpha_3|(1+s+r)+|\beta_3|\}.\end{aligned}$$

Let $0<\alpha_1\leqslant 1,\ \alpha_2\geqslant 0,\ \alpha_3\geqslant 0,\ \beta_i=0\ (i=1,2,3)$. Put also $1+(1-s+r)\alpha_2=1+(1+s-r)\alpha_3$, hence

$$\alpha_3=\frac{1-s+r}{1+s-r}\alpha_2. \tag{15}$$

By condition $f(\alpha)=1$ and (15) find

$$\alpha_2=\frac{1+s-r}{R}(1-\alpha_1),\qquad \alpha_3=\frac{1-s+r}{R}(1-\alpha_1).$$

For these values of α_i and β_i we get

$$B_1=|\alpha_1-1|(1+s+r)=(1+s+r)(1-\alpha_1),$$

$$B_2 = |\alpha_2|(1+s+r) = \frac{(1+s-r)(1+s+r)}{R}(1-\alpha_1),$$

$$B_3 = |\alpha_3|(1+s+r) = \frac{(1-s+r)(1+s+r)}{R}(1-\alpha_1).$$

The inequality $B_2 \geqslant B_3$ is equivalent to $1+s-r \geqslant 1-s+r$, which immediately follows from condition $s \geqslant r$. The inequality $B_2 \geqslant B_1$ is equivalent to $1 \leqslant \frac{1+s-r}{R}$, hence $(s-r)^2 + 2r \leqslant 1$. Since $s+r \leqslant 1$ it is sufficient to prove $(s-r)^2 + 2r \leqslant s+r$. From the last inequality we have $(s-r)^2 \leqslant s-r$, which immediately follows from $s-r \leqslant 1$ and $s \geqslant r$. Consequently, $\max_{1\leqslant i\leqslant 3} B_i = B_2$.

Moreover, $T_1 = 1 + (s+r-1)\alpha_1 \leqslant 1$, $T_2 = T_3 = 1 + \frac{(1-s+r)(1+s-r)}{R}(1-\alpha_1) \geqslant 1$, hence $\max_{1\leqslant i\leqslant 3} T_i = T_2$.

Now we will estimate k from the above. From the inequality $1 + kB_2 \leqslant T_2$ we easily get $k \leqslant \frac{1-s+r}{1+s+r}$. Obviously, $k \in (0, 1]$.

To show that $k = \frac{1-s+r}{1+s+r}$ is a maximal value of the constant of a strong unicity we prove that inequality

$$\begin{aligned} 1 + \frac{1-s+r}{1+s+r} \max\{ &|\alpha_1 - 1|(1+s+r) + |\beta_1|; \\ &|\alpha_2|(1+s+r) + |\beta_2|;\ |\alpha_3|(1+s+r) + |\beta_3|\} \\ \leqslant \max\{ &1 + (s+r-1)|\alpha_1| + |\beta_1|;\ 1 + (1-s+r)|\alpha_2| + |\beta_2|; \\ &1 + (1+s-r)|\alpha_3| + |\beta_3|\} \end{aligned} \tag{16}$$

satisfies for any α_i, β_i.

Consider three cases.

(1) Let $\max_i B_i = B_1$.

To prove (16) it is sufficient to show that

$$\begin{aligned} &1 + \frac{1-s+r}{1+s+r} \max\{|\alpha_1 - 1|(1+s+r) + |\beta_1|\} \\ &\leqslant \max\{1 + (1-s+r)|\alpha_2| + |\beta_2|;\ 1 + (1+s-r)|\alpha_3| + |\beta_3|\} \end{aligned} \tag{17}$$

Now we note that

$$|\alpha_1 - 1| = |s\alpha_2 + r\alpha_3| \leqslant s|\alpha_2| + r|\alpha_3|, \tag{18}$$

$$|\beta_1| = |-s\beta_2 - r\beta_3| \leqslant s|\beta_2| + r|\beta_3|, \tag{19}$$

$$\max\{1 + (1-s+r)|\alpha_2| + |\beta_2|;\ 1 + (1+s-r)|\alpha_3| + |\beta_3|\}$$
$$\geqslant \lambda_2(1 + (1-s+r)|\alpha_2| + |\beta_2|)$$
$$+ \lambda_3(1 + (1+s-r)|\alpha_3| + |\beta_3|), \qquad \text{where } \lambda_2 + \lambda_3 = 1.$$

Inequality (17) follows from

$$1 + (1-s+r)|\alpha_1 - 1| \leqslant \lambda_2(1 + (1-s+r)|\alpha_2|) + \lambda_3(1 + (1+s-r)|\alpha_3|) \tag{20}$$

$$\frac{1-s+r}{1+s+r}|\beta_1| \leqslant \lambda_2|\beta_2| + \lambda_3|\beta_3|. \tag{21}$$

Let, also as above, $\lambda_2 = \frac{s\mu}{1-s+r}$, $\lambda_3 = \frac{r\mu}{1+s-r}$, where $\mu = \frac{(1-s+r)(1+s-r)}{R}$. Then, by (18), we get

$$\lambda_2(1 + (1-s+r)|\alpha_2|) + \lambda_3(1 + (1+s-r)|\alpha_3|)$$
$$= 1 + \mu(s|\alpha_2| + r|\alpha_3|) \geqslant 1 + \mu|\alpha_1 - 1|.$$

Thus, to prove (20) it is sufficient to show that $1 + (1-s+r)|\alpha_1 - 1| \leqslant 1 + \mu|\alpha_1 - 1|$. The last inequality is equivalent to $(s-r)^2 + 2r \leqslant 1$, which is proved above.

Now we prove (21). By (19) we show that

$$\frac{1-s+r}{1+s+r}(s|\beta_2| + r|\beta_3|) \leqslant \frac{s(1+s-r)}{R}|\beta_2| + \frac{r(1-s+r)}{R}|\beta_3|.$$

The last inequality follows from

$$\frac{1-s+r}{1+s+r}|\beta_2| \leqslant \frac{1+s-r}{R}|\beta_2|, \tag{22}$$

$$\frac{1-s+r}{1+s+r}|\beta_3| \leqslant \frac{1-s+r}{R}|\beta_3|. \tag{23}$$

Inequality (22) follows from $1 - s + r \leqslant 1 + s - r$ and $R \leqslant 1 + s + r$. The first inequality is equivalent to $r \leqslant s$. The second inequality is equivalent to $(s - r)^2 \leqslant 1$.

Inequality (23) is equivalent to $R \leqslant 1 + s + r$.

(2) Let $\max_i B_i = B_2$.

It is sufficient to prove that

$1 + \frac{1-s+r}{1+s+r}(|\alpha_2|(1 + s + r) + |\beta_2|) \leqslant$
$1 + (1 - s + r)|\alpha_2| + |\beta_2|$, hence $\frac{1-s+r}{1+s+r}|\beta_2| \leqslant |\beta_2|$, then $k \leqslant 1$.

(3) Let $\max_i B_i = B_3$.

In this case it is sufficient to prove that

$$1 + \frac{1 - s + r}{1 + s + r}(|\alpha_3|(1 + s + r) + |\beta_3|) \leqslant 1 + (1 + s - r)|\alpha_3| + |\beta_3|.$$

The last inequality follows from $(1 - s + r)|\alpha_3| \leqslant (1 + s - r)|\alpha_3|$ and $k|\beta_3| \leqslant |\beta_3|$. The first inequality is equivalent to $r \leqslant s$. The second inequality is equivalent to $k \leqslant 1$. The proof is complete.

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